# Optimal Jamming against Digital Modulation

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***Abstract*—Jamming attacks can significantly impact the performance of wireless communication systems, and can lead to significant overhead in terms of re-transmissions and increased power consumption. This paper considers the problem of optimal jamming over an additive white Gaussian noise channel. We derive the optimal jamming signal for various digital amplitude-phase modulated constellations and show that it is not always optimal to match the jammer's signal to the victim signal in order to maximize the error probability at the victim receiver. Connections between the optimum jammer obtained in this analysis and the well-known pulsed jammer, popularly analyzed in the context of spread spectrum communication systems are illustrated. The gains obtained by the jammer when it knows the victim's modulation scheme and uses the optimal jamming signals obtained in this paper as opposed to conventional additive white Gaussian noise jamming are evaluated in terms of the additional signal power needed by the victim receiver to achieve same error rates under these two jamming strategies. We then extend these findings to obtain the optimal jamming signal distribution when a) the victim uses an OFDM-modulated signal and b) when there are multiple jammers attacking a single victim transmitter-receiver pair. Numerical results are presented in all the above cases to validate the theoretical inferences presented.**

## I. Introduction

The inherent openness of the wireless medium makes it susceptible to both intentional and un-intentional interference. Interference from neighboring cells in a wireless communication system is one of the major causes for un-intentional interference. On the flip side, intentional interference corresponds to adversarial attacks on a victim receiver that is not operating in a defensive mode. More generally, adversarial attacks in a wireless system can be broadly classified based on the capability of the adversary- a) *Eavesdropping attack*, in which the eavesdropper (passive adversary) can listen to the wireless channel and try to infer information (which if leaked may severely compromise data integrity) [1], [2], [3], b) *Jamming attack*, in which the jammer (active adversary) can transmit energy in order to disrupt reliable data transmission or reception [4], [5], [6] and c) *Hybrid attack*, in which an adversary operates "with the dual capability of either passively eavesdropping or actively jamming any ongoing transmission, with the objective of causing maximum disruption to the ability of the legitimate transmitter to share a secret message with its receiver" [7], [8].

In this paper, we study jamming attacks against practical wireless signals, namely digital amplitude-phase modulated signals. Jamming has traditionally been studied in the context of spread spectrum communications [9]. Barrage jamming, partial-band/narrow-band jamming, tone-jamming (where a victim is attacked by sending either a single or multiple jamming tones) and pulsed jamming are the most common types of jamming models considered in wireless communication systems. Deviating from these traditional simplistic techniques, we want to know "What is the optimum statistical distribution for power constrained jamming signals in order to maximize the error probability of digital amplitude-phase modulated constellations?" As will be discussed in detail shortly, this paper answers a question that is more relevant to practical wireless communication systems when compared to similar questions studied in the past, and consequently offers different solutions mainly because incorrect system models were previously considered and thus the wrong questions were answered.

Most of the earlier literature that asks similar questions regarding optimal jamming signals can be divided into two categories a) investigations that consider optimal jamming against Gaussian signaling schemes [4], [5], [6], [11], and b) investigations that study optimal jamming against pulse amplitude modulated (PAM) signals in the absence of ambient noise [12], [13]. Typically, an information theoretic framework was considered, e.g., [5], [6], [11], [13]. More specifically, in [5], the capacity of a wireless channel was analyzed in the presence of correlated jamming, where the authors showed that Gaussian signaling and Gaussian jamming form a saddle point solution. Here, the authors also showed that when the jammer does not have knowledge regarding the phase and the timing offsets introduced by the wireless channels, it is optimal in terms of the capacity minimization to be uncorrelated with the victim signal. In [6], [11], independent Gaussian input and noise (jamming signal) signals were again shown to be a saddle point solution for the mutual information game between the victim and the jammer. The convexity properties of error probability with respect to the AWGN jamming signal power against a binary-valued victim signal was studied in [12], which showed that a pulsed AWGN jamming signal is optimal. In [13], the worst case performance, in terms of maximizing the error probability and/or minimizing the capacity, achieved by any noise distribution was investigated when binary data was transmitted. The optimal noise distribution was shown to be a shifted version of the binary input signal.

Unfortunately, the previous works are not sufficiently realistic. This is because the works that belong to category (a) consider jamming against Gaussian signaling which is not used in practice and the works that belong to category (b) ignore the presence of thermal/ambient noise which is typically unavoidable in wireless environments. More importantly, these unrealistic assumptions lead to the wrong questions and thereby result in wrong conclusions, which are not relevant to practical wireless systems. The work closest to the present study is [14] which studies the optimal power distribution for any given jamming signal, but does not address the question raised in this paper. We show that optimal jamming signals against digital amplitude phase modulated constellations

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follow the statistical distribution of well-known modulation schemes under special conditions, and are not always matched (matching in this context refers to case where the jammer has the same distribution as the victim) to the victim signals. This finding is in contrast to the results obtained in the previous works.

In what follows, it is assumed that the victim receiver is not operating in an anti-jamming mode. Such jamming scenarios commonly occur because most transceivers do not employ jamming detection algorithms. For such a victim receiver, the decision regions for decoding the received signal will remain the same irrespective of the presence or absence of a jamming signal. For example, when the victim uses a symmetric binary signaling scheme i.e., signal levels given by $\pm A$ where $A$ is the amplitude of signaling, and is unaware of the presence of the jammer, the decision boundary will still be $0$. Improved jamming techniques, such as the ones proposed in this paper, help military and/or practical wireless communication systems jam their adversaries' received signal before they can interpret any sensitive information.

We assume in this work that the jammer is aware of the modulation scheme of the communication signal (for example, by employing modulation classification schemes [15], [16]) and also the power levels of the communication and the jamming signals at the victim receiver (using power control information, location and/or path loss calculations). These assumptions enable us to analyze the worst case jamming performance against standard modulation schemes. We first show that the optimal power-constrained jamming signal shares time only between two signal levels, *i.e.*, the jamming signal distribution takes the form of a binary distribution along any signaling dimension (in-phase and quadrature). Further, it will be shown that this binary distribution is nothing but the statistical distribution of well-known modulation schemes under special conditions, and that it is not always the same as the victim signal's distribution. These results are then extended to the more practical scenarios including (i) a non-coherent scenario where there is a phase mismatch between the jammers' signal and the victim signal, (ii) an asynchronous scenario where there is a timing offset between the jammers' signal and the victim signal and (iii) when the jammer is not perfectly aware of the power levels of the communication and jamming signals at the victim receiver.

We also extend this study to the cases where a) the victim transmitter-receiver pair use an OFDM-modulated signal to communicate and b) when multiple jammers attack a single victim transmitter-receiver pair. Most previous works that study jamming against OFDM signals consider AWGN jamming (see the tutorial paper [17] for more information), which as we show in this work is sub-optimal in terms of the error rates created at the victim receiver. Further, we show that under the same average power constraints i.e., the case where multiple jammers have the same total average power as a single jammer, significantly higher error rates can be achieved when the multiple jammers are perfectly coordinated.

The rest of this paper is organized as follows. The system model is introduced in Section II. The optimal jamming signal distribution when the victim and the jammers' signals are phase and time aligned is derived in Section III. In Section IV, the jammers' statistical distribution is derived for the cases when non-idealities are introduced by the wireless channel due to which the jamming and the victim signals are not perfectly aligned. In Section V, we extend the analysis to the case where the victim employs an OFDM modulated signal and in Section VI we study the performance of multiple jammers attacking a single victim transmitter-receiver pair. Numerical results are presented in sections III-VI to support the theoretical inferences made. Finally conclusions are drawn in Section VII.

## II. System Model

We assume that the data conveyed in the legitimate communication signal is mapped onto a known digital amplitude-phase constellation. The low pass equivalent of the transmitted signal is represented as $s(t) = \sum_{m=-\infty}^{\infty} \sqrt{P_S} s_m g(t - mT)$, where $P_S$ is the average received signal power, $g(t)$ is the real valued pulse shape and $T$ is the symbol interval. The complex random variables $s_m$ denote the modulated symbols, with a uniform distribution $f_S(s)$, i.e., all possible constellation points are equally likely. Without loss of generality, the average energy of $g(t)$ and modulated symbols $E(|s_m|^2)$ are normalized to unity.

It is assumed that the transmitted signal passes through an AWGN channel (received power is constant over the observation interval) while being attacked by a jamming signal represented as $j(t) = \sum_{m=-\infty}^{\infty} \sqrt{P_J} j_m g(t - mT)$, where $P_J$ is the average jamming signal power as seen at the victim receiver and $j_m$ denote the jamming symbols that are distributed according to $f_J(j)$ with $E(|j|^2) \leq 1$. Assuming a coherent receiver and perfect synchronization, the received signal after matched filtering and sampling once per symbol interval is given by

$$y_k = y(t = kT) = \sqrt{P_S} s_k + \sqrt{P_J} j_k + n_k, \ k = 1, 2, .. \quad (1)$$

where $n_k$ is zero-mean additive white Gaussian noise whose distribution is denoted by $f_N(n)$ with variance $\sigma^2$. The victim signal $s_k$, the jamming signal $j_k$ and the noise samples $n_k$ are all assumed to be statistically independent of each other. Let $\mathrm{SNR} = \frac{P_S}{\sigma^2}$ and $\mathrm{JNR} = \frac{P_J}{\sigma^2}$ indicate the signal power $(P_S)$ and jamming power $(P_J)$ to noise power $(\sigma^2)$ ratios respectively. In this paper, we initially assume that the jammer has perfect knowledge of $\mathrm{SNR}$ and $\mathrm{JNR}$ as seen at the victim receiver. This assumption allows for the analysis of the maximum error rates that can be created by the jammer at the victim receiver. We will later relax this assumption.

### A. Motivation

Here, we briefly motivate the reason to look beyond AWGN jamming. Consider a BPSK signaling scenario with $P_S = 1$, $P_J = 1$ and $\sigma^2 = 0$ (i.e. the channel does not add any noise). Thus the received signal is expressed as

$$y_k = s_k + j_k, \ k = 1, 2, \ldots . \quad (2)$$

If the jammer were aware of the signals sent by the transmitter, then it could negate them by sending the opposite of the

transmit signal, i.e., the jammer sends a $-1$ symbol to destroy a $+1$ symbol. However, this is not possible in real time as the jammer can not demodulate the transmit signal before transmission occurs. Hence, it sends a random BPSK signal to disrupt the communication. The receiver can decode the symbols correctly half of the time i.e., when it gets $\pm 2$. For the other half of the time when it gets $0$, it makes a random guess regarding the transmit signal with probability $\frac{1}{2}$ of being correct. Thus the overall error probability is $\frac{1}{4} = 0.25$. On the other hand, the error probability is $0.1587$ if an AWGN signal ($\sigma^2 = 1$) is used as the jamming signal [9]. For this toy example, the BPSK modulated jammer increased the error probability by $57.5\%$ as compared to the AWGN jammer (under similar power constraints) which suggests that there are interesting avenues to pursue beyond AWGN jamming.

## III. Perfect Channel Knowledge

First we analyze the statistics of the optimal jammer when it has perfect channel knowledge *i.e.*, the jamming signal is phase and time synchronous with the victim signal. In all the analysis that follows, it is assumed that the receiver is unaware of the presence of the jammer and hence the decision regions for the data detection remain the same as if there were no jammer. We first derive the optimum jamming signal distribution against a $M$-QAM[1] victim signal and later show that this can be simplified for specific modulation schemes.

For 2-dimensional signals, such as $M$-QAM, define $\bar{y}_k = [\Re y_k, \Im y_k]^T$ where $\Re y_k$ indicates the real (in-phase) part of $y_k$ and $\Im y_k$ indicates the imaginary (quadrature) part of $y_k$. Along similar lines we can define $\bar{s}_k, \bar{j}_k$ and $\bar{n}_k$ for all $k = 1, 2, \ldots, K$. Then (1) is rewritten as

$$\bar{y}_k = \sqrt{P_S}\bar{s}_k + \sqrt{P_J}\bar{j}_k + \bar{n}_k, \;\; k = 0, 1, \ldots, K. \tag{3}$$

Since the victim signal and the jammer's signal are coherent, 2-dimensional modulation schemes such as $M$-QAM can be analyzed by considering them as two independent $\sqrt{M}$-PAM signals along the in-phase and quadrature dimensions [22]. Therefore, the average probability of error[2] $p_e$ of an $M$-QAM victim signal along any signaling dimension in an AWGN channel in the presence of jamming signal $\bar{j}$ is given by

$$p_e(j, \mathrm{SNR}, \mathrm{JNR}) \approx \left(1 - \frac{1}{\sqrt{M}}\right)\frac{1}{2}\times \left[\mathsf{erfc}\left(\sqrt{\mathrm{SNR}}\frac{d_{\mathsf{min}}}{2} + \sqrt{\mathrm{JNR}}j\right) + \mathsf{erfc}\left(\sqrt{\mathrm{SNR}}\frac{d_{\mathsf{min}}}{2} - \sqrt{\mathrm{JNR}}j\right)\right], \tag{4}$$

where $j = \Re\bar{j}$ or $\Im\bar{j}$, $M$ is the order of the constellation and $d_{\mathsf{min}}$ is the minimum distance of the underlying modulation scheme [22].

The jammer intends to maximize this error probability by transmitting a sequence of symbols $j$ (along the in-phase and the quadrature dimensions) which are to be chosen based on $P_S$ (or SNR) and $P_J$ (or JNR). Notice that (4) is symmetric in $j$. Therefore, $p_e(j, \mathrm{SNR}, \mathrm{JNR}) = p_e(-j, \mathrm{SNR}, \mathrm{JNR}) = p_e(|j|, \mathrm{SNR}, \mathrm{JNR})$. Hence, the polarity of $j$ is irrelevant here. Therefore, the error probability is maximized over the distribution of $a = |j|$. However, to maximize the entropy of the jamming signal, transmitting a value of $a$, implies transmitting $j = +a$ and $j = -a$ with equal probability. The optimization problem for such a jammer can thus be formulated as

$$\begin{aligned} &\max_{f_A} \int_a p_e\left(a, \mathrm{SNR}, \mathrm{JNR}\right) f_A(a) da \;\; \mathsf{s.t.}\; E(a^2) \leq \frac{1}{2}, \\ &\equiv \max_{f_A} E\Big(p_e(a, \mathrm{SNR}, \mathrm{JNR})\Big) \;\; \mathsf{s.t.}\; E(a^2) \leq \frac{1}{2}. \end{aligned} \tag{5}$$

Notice that the optimization is over the signal level distribution $f_A$ and that $E(a^2) \leq \frac{1}{2}$ because we consider only one signaling dimension (recall that $E(||\bar{j}||^2) \leq 1$, where $||\bar{j}||$ indicates the norm of the vector $\bar{j}$).[3] Similar optimization problems have previously been studied in the context of stochastic signaling for maximizing the probability of signal detection and minimizing the error probability in [14], [18], [19] and references therein. Below, we briefly present the solution for the optimization problem in (5). A more elaborate and general proof for this optimization problem will be shown in Theorem 4 in Section VI, where we discuss the case of multiple jammers attacking a single victim receiver.

### A. Optimum Jamming Signal Distribution

Define sets $U$ and $W$ as

$$U = \{(u_1, u_2) : u_1 = p_e\left(a, \mathrm{SNR}, \mathrm{JNR}\right), u_2 = a^2\}$$
$$W = \{(w_1, w_2) : w_1 = E_{f_A}\Big(p_e\left(a, \mathrm{SNR}, \mathrm{JNR}\right)\Big), w_2 = E_{f_A}(a^2)\}.$$

Since $p_e\left(a, \mathrm{SNR}, \mathrm{JNR}\right)$ is a continuous function ($\mathsf{erfc}$ is a continuous function) defined on the support of $a$, the mapping from $[0, a_{\max}]$ (notice that it can be safely assumed that $a \leq a_{\mathsf{max}}$ for some finite $a_{\mathsf{max}} > 0$ since arbitrarily large amplitudes of the signal cannot be generated by any practical transmitter) to $(\mathbb{R}^+)^2$ defined by $\left(p_e\left(a, \mathrm{SNR}, \mathrm{JNR}\right), a^2\right)$ is continuous. Since the continuous image of a compact set is compact, the set $U$ is also compact.

Since $U$ is compact, the convex hull $V$ of $U$ is closed with dimensions smaller than or equal to 2 because $U$ and $V$ are subsets of $(\mathbb{R}^+)^2$. Based on the definition of the set $W$, it can be shown that $V = W$ [18, Proposition 3], [20] (an elaborate proof for this is given in Theorem 4). Further, Carathéodory's theorem [21], states that any point in $V$ can be expressed as a convex combination of at most three points in $U$ as they belong to $(\mathbb{R}^+)^2$. Since the optimal jamming signal pdf should maximize the objective function, the optimal solution exists on the boundary i.e., on $V$ (as it is a closed set). Since any point on the boundary can be expressed as a convex combination of at most 2 elements in $U$, the optimal jamming signal level pdf $f_A(a)$ can be represented as a discrete random variable with at most 2 mass points.

[1]The analysis presented in this paper can also be extended to cross-QAM signals by using the appropriate error probability expressions. But in this paper, we focus on square QAM signals.

[2]With a slight abuse of notation, we use $p_e$ to denote the probability of error. The variables that it depends on are shown within brackets. For example, $p_e(P_S)$ indicates that $p_e$ is a function of the signal power $P_S$.

[3]Depending on whether the victim's modulation scheme is symmetric or not along the in-phase and quadrature dimensions, this constraint can be changed accordingly.

$$p_e(\lambda,a_1,a_2,\mathrm{SNR},\mathrm{JNR}) \approx \left(1-\frac{1}{\sqrt{M}}\right)\frac{1}{2}\Bigg\{\lambda\left[\mathsf{erfc}\left(\sqrt{\mathrm{SNR}}\frac{d_{\mathsf{min}}}{2}+\sqrt{\mathrm{JNR}}a_1\right)+\mathsf{erfc}\left(\sqrt{\mathrm{SNR}}\frac{d_{\mathsf{min}}}{2}-\sqrt{\mathrm{JNR}}a_1\right)\right] + (1-\lambda)\left[\mathsf{erfc}\left(\sqrt{\mathrm{SNR}}\frac{d_{\mathsf{min}}}{2}+\sqrt{\mathrm{JNR}}a_2\right)+\mathsf{erfc}\left(\sqrt{\mathrm{SNR}}\frac{d_{\mathsf{min}}}{2}-\sqrt{\mathrm{JNR}}a_2\right)\right]\Bigg\} \quad (7)$$

Since the above proof is generic for both the in-phase and quadrature dimensions, the optimal jamming signal distribution has at most two signal levels along any signaling dimension. Therefore the optimal jamming signal pdf along any signaling dimension is given by

$$f_A(a) = \lambda\delta(a-a_1)+(1-\lambda)\delta(a-a_2),\ \lambda\in[0,1]$$
$$\lambda a_1^2+(1-\lambda)a_2^2 \le \frac{1}{2}, \quad (6)$$

where $\lambda$ and $(1-\lambda)$ are the probabilities with which the jammer sends signals $a_1$ and $a_2$ respectively along any dimension and $\delta(a)$ is the Dirac-delta function. Thus, the problem of finding an optimum jamming signal distribution is now reduced to finding $\lambda$, $a_1$ and $a_2$ rather than a continuous distribution $f_J(\bar{j})$ for the jamming signal $\bar{j}$ given in (3).

***Remark** 1:* It is important to notice that this analysis holds true for any $M$-PAM and $M$-QAM signals because we started the analysis with the $p_e$ of an $M$-QAM signal by decomposing it into two $\sqrt{M}$-PAM signals. Appropriate $p_e$ expressions and average jamming signal energy constraints must be used based on the victim signal's modulation. For example, the average jamming signal energy constraint $E(a^2)\le 1$ for the case of $M$-PAM signals as it is natural to consider jamming signals only in the in-phase dimension against $M$-PAM signals (only then $p_e$ will be maximized) and $E(a^2)\le\frac{1}{2}$ for two-dimensional signals such as $M$-QAM.

### *B. Analysis against $M$-QAM victim signals*

For the case of jamming against a $M$-QAM victim signal, it is not hard to show that $p_e(a,\mathrm{SNR},\mathrm{JNR})$ in (4) is a non-decreasing function of $a$ and hence $p_e$ is maximized on the boundary defined by $E(a^2)=1/2$ (it is $1/2$ because we consider only one signaling dimension). Using the fact that the optimum jamming signal level distribution is given by (6), the overall $p_e$ along any signaling dimension is given by (7).

Numerically obtaining the optimal jamming signal distribution under an average power constraint is difficult for a general range of SNR and JNR. Similar optimization problems have been solved using global optimization techniques such as particle swarm optimization in [18]. In this paper, we first state 3 theorems that help in establishing the optimal jamming signal distribution against $M$-QAM victim signals for certain ranges of SNR and JNR. Due to a lack of space, we only sketch the proofs of these theorems. Later, we support these claims and present the optimal jamming signals for a general range of SNR and JNR via simulations performed using the optimization toolbox in Matlab. We also present several remarks that help the exposition of these Theorems easier.

***Theorem** 1:* QPSK is the optimal jamming signal when the victim signal uses $M$-QAM and $\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}}<\sqrt{\mathrm{JNR}}\,\tanh\left(2\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}\mathrm{JNR}}\right)$.

***Remark** 2: Notice that* $\tanh\left(2\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}\mathrm{JNR}}\right)\approx 1$ *when* $\mathrm{SNR}\frac{d_{\min}^2}{2}\mathrm{JNR}>1$. *Therefore, in this case, QPSK is the optimal jamming signal when* $\mathrm{SNR}\frac{d_{\min}^2}{2}<\mathrm{JNR}$ *which is a stricter condition than* $\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}}<\sqrt{\mathrm{JNR}}\,\tanh\left(2\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}\mathrm{JNR}}\right)$.

***Remark** 3: The theoretical $p_e$ when QPSK is used as a jamming signal is given by substituting $a_1=\frac{1}{\sqrt{2}}$ and $\lambda=1$ in* (7). *The slope of the error probability with respect to* SNR *i.e.,* $\frac{\partial p_e}{\partial \mathrm{SNR}}$ *within a proportionality constant when* $\mathrm{SNR}\frac{d_{\min}^2}{2}<\mathrm{JNR}$ *and* $\mathrm{SNR}\frac{d_{\min}^2}{2}\mathrm{JNR}>1$ *can be approximated as*

$$\text{AWGN:}\frac{-1}{\sqrt{\mathrm{SNR}\times\mathrm{JNR}}};\ \text{QPSK:}\frac{-2}{\sqrt{\mathrm{SNR}\times\exp(\mathrm{JNR})}}, \quad (8)$$

*which shows that the error probability due to a QPSK jamming signal decays more slowly with* JNR *when compared to the AWGN jamming signal. Thus from a jammers' perspective it is advantageous to use a QPSK jamming signal when compared to traditional AWGN jamming.*

**Proof**: Since $E(a^2)=\frac{1}{2}$, we have $\lambda a_1^2+(1-\lambda)a_2^2=\frac{1}{2}$. Using this relationship, (7) can be written as a function of $a_1$ denoted by $p_e(\lambda,a_1,\mathrm{SNR},\mathrm{JNR})$. $a_1=\{0,\frac{1}{\sqrt{2\lambda}},\frac{1}{\sqrt{2}}\}$ are the solutions to $\frac{\partial p_e(\lambda,a_1,\mathrm{SNR},\mathrm{JNR})}{\partial a_1}=0$. To prove the optimality of QPSK, we need to show that $p_e(\lambda,a_1,\mathrm{SNR},\mathrm{JNR})$ is maximized at $a_1=\frac{1}{\sqrt{2}}$ (we will discuss the solutions $a_1=0,\frac{1}{\sqrt{2\lambda}}$ in Theorems 2 and 3). The second derivative of $p_e$ with respect to $a_1$ i.e., $\frac{\partial^2 p_e(\lambda,a_1,\mathrm{SNR},\mathrm{JNR})}{\partial a_1^2}\big|_{a_1=\frac{1}{\sqrt{2}}}$ has 4 terms, each of which can be shown to be $<0$ when $\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}}<\sqrt{\mathrm{JNR}}\,\tanh\left(2\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}\mathrm{JNR}}\right)$. Further, this holds true irrespective of $a_2$ when $a_1=\frac{1}{\sqrt{2}}$ and $\lambda=1$. This case is still in agreement with the optimal jamming signal distribution because the proof of the optimal jamming signal pdf only says that the optimal jamming signal distribution can be represented by a randomization of *at most* two different signal levels. Also recall that $a=|\Re(\bar{j})|=|\Im(\bar{j})|$. Thus, $\Re(\bar{j})=\Im(\bar{j})=\pm\frac{1}{\sqrt{2}}$.

From [23], it is well known that the entropy of any two-point distribution is maximized when each of the points is equally likely. Therefore the jammer sends a $\frac{1}{\sqrt{2}}$ symbol on the in-phase or the quadrature dimension with probability $\frac{1}{2}$ and $-\frac{1}{\sqrt{2}}$ also with probability $\frac{1}{2}$. Therefore the optimal jamming signal distribution is QPSK when $\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}}<\sqrt{\mathrm{JNR}}\,\tanh\left(2\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}\mathrm{JNR}}\right)$.

$$\sqrt{2\mathrm{SNR}d_{\min}^2\mathrm{JNR}}\exp\left(-\frac{\mathrm{SNR}d_{\min}^2}{2}\right) < \sqrt{1-\lambda_{opt}}\left\{\exp\left(-\left(\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}}-\sqrt{\frac{\mathrm{JNR}}{(1-\lambda_{opt})}}\right)^2\right)\right.$$
$$\left.-\exp\left(-\left(\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}}+\sqrt{\frac{\mathrm{JNR}}{(1-\lambda_{opt})}}\right)^2\right)\right\}, \quad (9)$$

***Theorem** 2:* $\{a_1, a_2\} = \left\{0, \frac{1}{\sqrt{2(1-\lambda_{opt})}}\right\}$ is the optimal jamming signal along any signaling dimension when $\frac{\partial p_e(\lambda,a_1,\mathrm{SNR},\mathrm{JNR})}{\partial a_1}|_{a_1=0} = 0$ and $\frac{\partial^2 p_e(\lambda,a_1,\mathrm{SNR},\mathrm{JNR})}{\partial a_1^2}|_{a_1=0} < 0$ i.e., $p_e$ has a maxima at $a_1 = 0$. $\lambda_{opt}$ is obtained by solving for $\lambda$ in the equation $\frac{\partial p_e(\lambda,a_1,\mathrm{SNR},\mathrm{JNR})}{\partial \lambda}|_{a_1=0} = 0$ and also satisfying (9).

**Proof**: As mentioned earlier, $a_1$=0 is a solution for $\frac{\partial p_e(\lambda,a_1,\mathrm{SNR},\mathrm{JNR})}{\partial a_1} = 0$. When this is true, the optimal value of $\lambda$ denoted by $\lambda_{opt}$ is obtained by solving $\frac{\partial p_e(\lambda,a_1,\mathrm{SNR},\mathrm{JNR})}{\partial \lambda}|_{a_1=0} = 0$. Further, it can be proved that $\frac{\partial^2 p_e(\lambda_{opt},a_1,\mathrm{SNR},\mathrm{JNR})}{\partial a_1^2}|_{a_1=0}$ will be $< 0$ only when $\lambda_{opt}$ satisfies (9). By symmetry $\left\{\frac{1}{\sqrt{2\lambda_{opt}}}, 0\right\}$ is also a solution. Such a solution is also known as on-off keying since the jammer sends power using only one of the two possible signal levels, either $a_1$ or $a_2$. Such a jamming signal will help to increase the $p_e$ in cases where the jammer has limited power. The significance of this solution will be explained in more detail shortly.

***Remark** 4: When on-off keying is optimal, it can be shown that $p_e$ is equivalent to the probability of error achieved when the jammer uses QPSK signaling and either transmits with power $\frac{P_J}{\lambda_{opt}}$ or shuts off transmission with probability $\lambda_{opt}$ and $(1-\lambda_{opt})$ respectively. Such a jamming signal is equivalent to a pulsed jammer albeit modulated by a QPSK signal rather than AWGN [9], [24]. Exploiting this equivalence, we next explicitly characterize the range of* SNR *and* JNR *where on-off keying/pulsing is optimal.*

***Theorem** 3:* For a given SNR and JNR, the optimum strategy for a QPSK modulated jammer is to use two different power levels, one of which is 0, when $\mathrm{JNR} \leq \widehat{\mathrm{JNR}}$, and each of these power levels is used with a probability depending on the victim signal parameters. However, if $\mathrm{JNR} > \widehat{\mathrm{JNR}}$ then the optimal strategy is to employ continuous jamming. $\widehat{\mathrm{JNR}}$ is defined by a unique jamming signal power $\widehat{P_J}$ and signal power $P_S$ such that $p_e$ is convex when $\mathrm{JNR} \leq \widehat{\mathrm{JNR}}$ and concave elsewhere. In other words, $\widehat{P_J}$ is an inflection point for $p_e$. Henceforth, such a jamming signal will be referred to as the pulsed-jammer.

**Proof**: When QPSK is used as the jamming signal, it can be shown that for the $p_e$ in (4), there exists a single inflection point $\widehat{\mathrm{JNR}}$ such that $p_e$ is convex when $\mathrm{JNR} \leq \widehat{\mathrm{JNR}}$ and concave elsewhere. When $p_e$ is convex, the error probability can be increased by time sharing between two different power levels (by Jensen's inequality) [23] with probability $\rho$ and under the constraint that the average power is still $P_J$. Then,

TABLE I
OPTIMAL JAMMING SIGNAL LEVEL DISTRIBUTION AGAINST A 16-QAM VICTIM SIGNAL, $\mathrm{JNR} = 10\ dB$.
$a_1, a_2$ INDICATE THE ABSOLUTE VALUES OF THE REAL AND IMAGINARY PARTS OF THE JAMMING SIGNAL.

| SNR | $\lambda_{opt}$ | $a_1$ | $a_2$ |
|---|---|---|---|
| -2 | 1 | $\frac{1}{\sqrt{2}}$ | n/a |
| 4 | 1 | $\frac{1}{\sqrt{2}}$ | n/a |
| 10 | 1 | $\frac{1}{\sqrt{2}}$ | n/a |
| 16 | 1 | $\frac{1}{\sqrt{2}}$ | n/a |
| 19 | 0.4633 | 1.039 | 0 |
| 22 | 0.2304 | 1.473 | 0 |
| 25 | 0.1193 | 2.047 | 0 |
| 28 | 0.0625 | 2.828 | 0 |

the achievable $p_e$ when $\mathrm{JNR} \leq \widehat{\mathrm{JNR}}$ is given by

$$\rho p_e\left(1, \frac{1}{\sqrt{2}}, \mathrm{SNR}, \frac{\mathrm{JNR}}{\rho}\right) + (1-\rho)p_e\left(1, \frac{1}{\sqrt{2}}, \mathrm{SNR}, 0\right), \quad (10)$$

where, the optimal value of $\rho$ can be found by using the first and second derivatives of (10). Since (10) is equivalent to the $p_e$ obtained in Theorem 2, it is not hard to see that the optimal value of $\rho$ is given by $\lambda_{opt}$ (discussed in Theorem 2). When $\mathrm{JNR} \geq \widehat{\mathrm{JNR}}$ i.e., the concave region, the achievable $p_e$ is described by $p_e\left(1, \frac{1}{\sqrt{2}}, \mathrm{SNR}, \mathrm{JNR}\right)$ (the QPSK jamming case) which is indicative of continuous jamming. This concludes the proof of the theorem.

Table III-B shows the optimal values of the three unknown parameters against a 16-QAM victim signal for a general case in which $\mathrm{SNR} \gtreqless \mathrm{JNR}$ ($a_1, a_2$ are defined earlier). Since the in-phase and quadrature dimensions are equivalent in such a coherent scenario, it is seen that the optimal jamming signal is the same along any signaling dimension as mentioned earlier.

***Remark** 5: From Table III-B, it can be seen that QPSK is the optimal jamming signal only until a certain* SNR *beyond which a pulsed-QPSK jammer is optimal. Further, the pulsing duration decreases as the* SNR *increases, which indicates that the jammer is transmitting only for a fraction of the time. However, it jams the receiver with increased signal levels in an attempt to compensate for the increased* SNR*. Notice that the pulsed jamming signal is not a shifted version of the input signal which is different from the results in [13] because we have considered the effects of the additional AWGN noise introduced by the channel. Further, it is easy to show that the sign detector (for example, in the case of binary inputs) is no longer the maximum likelihood detector when the jamming signal is obtained by using the foregoing analysis. This finding is again in contrast to the conclusions made in [13], mainly because the additional AWGN noise introduced by the channel*

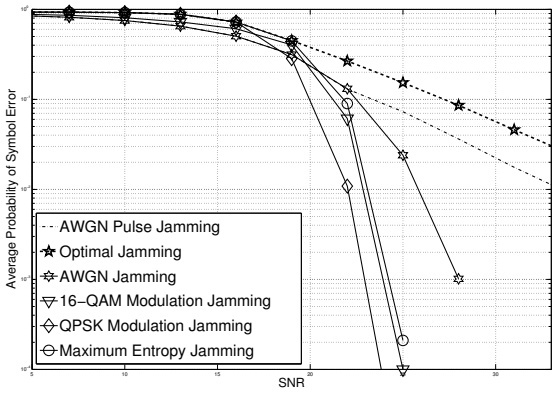


Fig. 1. Comparison of various jamming techniques against a 16-QAM modulated victim signal, $\mathrm{JNR} = 10\ dB$.

*is also considered in the present work. However, notice that the optimal maximum likelihood detection at the victim receiver in the presence of jamming is not the focus of this present work.*

The $p_e$ for the 16-QAM victim signal under various jamming scenarios is shown in Fig. 1. Here 16-QAM (QPSK) jamming refers to a randomly generated 16-QAM (QPSK) modulated jamming signal, and AWGN jamming refers to a zero-mean white Gaussian noise jamming signal with variance $P_J$. It is well known that the entropy of a two-level distribution is maximized when $\lambda = 0.5$ [23]. A new optimization problem (extending the one in (5) and (6)) is solved by introducing an additional constraint where $\lambda = 0.5$. We call such a jamming scenario as *maximum entropy jamming*. While *maximum entropy jamming* is better than QPSK jamming, it is worse than the optimal jamming as the constraint $\lambda = 0.5$ does not allow the optimization algorithm to explore the pulsed jamming solution. For a fair comparison, the jamming performance of a pulsed jammer modulated with an AWGN signal [9] is also shown in Fig. 1. The optimal pulsing ratio $\lambda_{opt}^{\mathrm{AWGN}}$ for the pulsed AWGN jamming signal against any $M$-QAM victim signal is obtained by using the first and second derivatives with respect to $\lambda$ of

$$\left(1-\frac{1}{\sqrt{M}}\right)\left[\lambda \mathsf{erfc}\left(\sqrt{\frac{\mathrm{SNR}}{(1+\frac{\mathrm{JNR}}{\lambda})}}\frac{d_{\mathsf{min}}}{2}\right)+(1-\lambda)\mathsf{erfc}\left(\sqrt{\mathrm{SNR}}\frac{d_{\mathsf{min}}}{2}\right)\right].$$

AWGN-based pulsed jamming converts the exponential relationship between $p_e$ and $\mathrm{SNR}$ to a linear one [9]. This also holds true for the case of the optimal jamming as seen in Fig. 1. This is similar to the behavior of $p_e$ in a Rayleigh fading channel where it is inversely proportional to $\mathrm{SNR}$. Intuitively, a symbol erased due to a deep fade is similar to the case where a symbol is disrupted by jamming. Thus, the optimal jammer is capable of generating a fading channel-like scenario in an AWGN channel.

In summary, we first showed that the optimal jamming signal distribution has a discrete distribution with only two mass points along any signaling dimension. Specifically, we showed that the signal levels of the in-phase or quadrature parts of the jamming signal will obey a two-point distribution. Using this result to address jamming against a $M$-QAM victim signal, we presented three Theorems, where Theorem 1 shows that QPSK is the optimal jamming signal against a $M$-QAM victim signal when $\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}} < \sqrt{\mathrm{JNR}}\ \tanh\left(2\sqrt{\mathrm{SNR}\frac{d_{\min}^2}{2}\mathrm{JNR}}\right)$, Theorems 2 and 3 show that pulsed QPSK is an optimal jamming signal when $\mathrm{JNR} \leq \widehat{\mathrm{JNR}}$. We used numerical optimization techniques to obtain a solution over all $\mathrm{SNR}, \mathrm{JNR}$, based on which it is conjectured that *pulsed-QPSK* is the optimal signal to jam any $M$-QAM modulated victim signal.

TABLE II
OPTIMAL JAMMING SIGNALS IN A COHERENT SCENARIO.

| Victim signal | Modulation scheme of pulsed jamming signal |
|---|---|
| BPSK | BPSK |
| QPSK | QPSK |
| 4-PAM | BPSK |
| 16-QAM | QPSK |

***Remark** 6: Since the two dimensional $M$-QAM constellations were analyzed by treating them as two orthogonal $\sqrt{M}$-PAM signals, the above analysis can be directly simplified for one dimensional signaling constellations such as BPSK, 4-PAM among others. Table II summarizes the optimal jamming signals against commonly used digital amplitude-phase modulated constellations. These results indicate that matching the jamming signal to the victim signal i.e., using the same signal as the victim is not always optimal.*

## IV. FACTORS THAT MITIGATE JAMMING

In this section, jamming is studied when the victim signal is not coherent (i.e., phase or time asynchronous) with the jamming signal when they reach the victim receiver. From a jammers' perspective, these non-idealities in the channel, specifically differences between the victim and jamming signals will lower the impact of jamming at the victim receiver. For example, consider a scenario where the victim signal uses BPSK and the jammer also sends a BPSK signal. If the channel introduces a $90^\circ$ phase offset between these two signals, then the jammers' signal does not have any impact on the victim signal (as the receiver only demodulates the projections of the signal received along the in-phase dimension).

If the phase/time shift between the victim signal and the jamming signal is known ahead of time to the jammer, it can compensate for this in the jamming signal before it is sent. However, this may be difficult to achieve in a real time communication system. Hence, in this section, we consider scenarios where the jammer is unaware of (or unable to compensate for) this random phase or time offset introduced by the wireless channel and thus treats it as a random variable. From a jammers' perspective it is necessary to optimize its signal distribution across all random offsets introduced by the channel. We also consider the case when the jammer is not perfectly aware of the communication and jamming signal power levels as seen at the victim receiver.

### A. Non-Coherent Jamming

In this sub-section, jamming behavior is studied when the jammers' signal is not coherent (i.e., phase asynchronous) with

$$p_e(\lambda, \bar{j}, \mathrm{SNR}, \mathrm{JNR}) \approx \left(1 - \frac{1}{\sqrt{M}}\right)\frac{1}{2}\Bigg[\mathsf{erfc}\left(\sqrt{\mathrm{SNR}}\frac{d_{\mathsf{min}}}{2} + \sqrt{\mathrm{JNR}}(\Re\bar{j}\cos(\phi) - \Im\bar{j}\sin(\phi))\right) + \mathsf{erfc}\left(\sqrt{\mathrm{SNR}}\frac{d_{\mathsf{min}}}{2} - \sqrt{\mathrm{JNR}}(\Re\bar{j}\cos(\phi) - \Im\bar{j}\sin(\phi))\right)\Bigg]. \quad (13)$$

TABLE III
OPTIMAL NON-COHERENT JAMMING SIGNAL LEVEL DISTRIBUTION AGAINST A 16-QAM VICTIM SIGNAL, $\mathrm{JNR} = 10\,dB$. $a_1, a_2$ INDICATE THE ABSOLUTE VALUES OF THE REAL AND IMAGINARY PARTS OF THE JAMMING SIGNAL.

| SNR | $\lambda_{opt}$ | $a_1$ | $a_2$ |
|---|---|---|---|
| -2 | 1 | $\frac{1}{\sqrt{2}}$ | n/a |
| 4 | 1 | $\frac{1}{\sqrt{2}}$ | n/a |
| 10 | 1 | $\frac{1}{\sqrt{2}}$ | n/a |
| 16 | 1 | $\frac{1}{\sqrt{2}}$ | n/a |
| 19 | 1 | $\frac{1}{\sqrt{2}}$ | n/a |
| 22 | 0.185 | 1.642 | 0 |
| 25 | 0.094 | 2.304 | 0 |
| 28 | 0.049 | 3.211 | 0 |

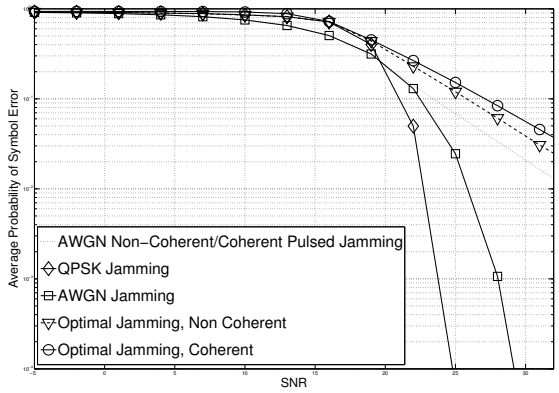


Fig. 2. Comparison of jamming techniques against a 16-QAM victim signal in a non-coherent (random phase offset) scenario, $\mathrm{JNR} = 10\,dB$.

the victim signal. With a random phase offset, the victim signal is given by

$$\bar{y}_k = \sqrt{P_S}\bar{s}_k + \sqrt{P_J}\mathsf{exp}\,(i\phi)\,\bar{j}_k + \bar{n}_k,\ k = 0, 1, \ldots, K, \quad (11)$$

where $\phi$ indicates the phase offset between the victim signal and the jamming signal at the victim receiver and is treated as a uniform random variable between 0 and $2\pi$, and $i = \sqrt{-1}$. As in Section III, the optimization problem for the jammer is given by

$$\max_{f_{\bar{j}}}\ E_{f_{\bar{j}}}\Big[E_{\phi}\Big(p_e\left(\bar{j}, P_S, P_J\right)\Big)\Big]\ \text{s.t.}\ E(||\bar{j}||^2) \leq 1. \quad (12)$$

The optimal jamming signal distribution along any signaling dimension in the non-coherent scenario is obtained by following the analysis in Section III and is described below. Without loss of generality, we present the analysis for jamming against a $M$-QAM victim signal as done in Section III.

Even in the non-coherent case, the $M$-QAM signal can be analyzed by considering it as two orthogonal $\sqrt{M}$-PAM signals. However, in this case due to the random phase offset between the jammers' signal and the victim signal, projections of the jammers' signal along each signaling dimension must be considered which is different from the analysis in Section III. The $p_e$ of a $M$-QAM signal along the in-phase dimension when there is a jamming signal $\bar{j}$ and a random phase offset $\phi$, is given by (13). A similar expression holds true for the quadrature signaling dimension. Using (13) and solving the optimization problem in (12) by following the analysis in Section III gives the optimal jamming signal level distribution shown in Table IV-A against a 16-QAM victim signal (we used the numerical optimization toolbox in Matlab to solve the optimization problem). Even in this case $a_1, a_2$ indicate the possible absolute values of the real and imaginary parts of the jamming signal $\bar{j}$.

It is interesting to see that once again QPSK (recall that in the coherent scenario, the optimal jamming signal distribution along any signaling dimension was defined by $\pm a_1$ where $a_1$ and $-a_1$ are both transmitted with equal probability, the same definition holds true even in the non-coherent scenario) is the optimal jamming signal until a certain SNR. Beyond this limit, pulsed-QPSK is the optimal jamming signal. This behavior is similar to the observations in Section III. When pulsing is optimal, the non-zero signal level is given by its corresponding probability as $\frac{1}{\sqrt{2\lambda_{opt}}}$ or $\frac{1}{\sqrt{2(1-\lambda_{opt})}}$, which in other words means that a QPSK modulated jammer with pulsing ratio $\lambda_{opt}$ is the optimal jamming signal. In Table IV-A, notice that the optimization solver returned equal values for the jamming signal levels along the in-phase and quadrature dimensions. This is due to the symmetry along these dimensions in the case of 2-dimensional signaling which holds true irrespective of whether the jamming signal is coherent or non-coherent with the victim signal.

Similar to the coherent scenario (see Theorem 3), pulsed-QPSK can be shown to be optimal when $\mathrm{JNR} \geq \widehat{\mathrm{JNR}}$ where $\widehat{\mathrm{JNR}}$ is the inflection point of $p_e$ with respect to JNR. As the presence of a phase offset reduces the jamming effect, $\widehat{\mathrm{JNR}}$ in the non-coherent case is higher when compared to $\widehat{\mathrm{JNR}}$ in a coherent scenario. In other words, for a given JNR the SNR at which pulsing is optimal increases (as seen in Table IV-A) in a non-coherent scenario in comparison to the coherent scenario which was discussed earlier. The performance of the various jamming signals against a 16-QAM victim signal is shown in Fig. 2. Although the $p_e$ achieved by the optimal jamming signal (or pulsed-QPSK) is less when compared to the coherent scenario (due to phase mismatch), it is still higher than the $p_e$ achieved using pulsed-AWGN jamming.

Similar to the coherent scenario, the analysis for the $M$-QAM constellations can be extended to any specific modula-

tion scheme in a non-coherent scenario. The optimal jamming signals in such a phase asynchronous scenario against the commonly used modulation schemes such as BPSK, 4-PAM, QPSK and 16-QAM are still given by Table II.[4] However, the pulsed jamming duration of these optimal jamming signals changes between the coherent and non-coherent (phase asynchronous) scenarios as seen from Tables III-B and IV-A. As seen from Fig. 2, the gain in the SNR required to achieve a target $p_e$ when compared to the coherent scenario, decreases by 1-2 $dB$ due to this phase mismatch.

### B. Symbol Timing Offset

Similar to the phase offset, although the jammer identifies the modulation scheme and is aware of the symbol interval used by the victim signal, it is difficult to be time aligned due to the unknown delays introduced by the wireless channel. In this sub-section we consider the jamming performance when the victim signal and the jammers' signal are not time synchronized with each other. Note that, in general, phase offset and symbol timing offset can occur together in practical wireless communication systems, but we do not consider both these non-idealities together in this paper due to the complexity involved in optimizing the jamming signal. However, the framework developed thus far is still applicable and can be extended to such complex scenarios. Below, we focus on the effects of timing offset on the jamming performance.

The low pass equivalent of the jammed victim signal after matched filtering is given by

$$y_k = \sqrt{P_S}s_k + \sqrt{P_J}\sum_{m=-\infty}^{\infty} j_m \widehat{g}(t - mT - \tau) + n_k, \quad k = 1, 2, \ldots, \quad (14)$$

where $\tau$ indicates the symbol timing offset/random delay introduced by the channel, $\widehat{g}(t)$ is a Nyquist pulse at the output of the matched filter given by $\widehat{g}(t) = g(t) * g(t)$, where $*$ indicates the convolution operation. For 2-dimensional signals, such as $M$-QAM, let $\bar{y}_k = [\Re y_k, \Im y_k]^T$ and along similar lines define $\bar{s}_k, \bar{j}_k$ and $\bar{n}_k$ for all $k = 1, 2, \ldots, K$. Then (14) is rewritten as

$$\bar{y}_k = \sqrt{P_S}\bar{s}_k + \sqrt{P_J}\sum_{m=-\infty}^{\infty} \bar{j}_m \widehat{g}(t - mT - \tau) + \bar{n}_k, \quad k = 1, 2, \ldots, . \quad (15)$$

By taking the pulse shape $\widehat{g}(t)$ to be zero for $|t| \geq MT$ (in fact, real implementations must truncate these pulses), the samples $\{y_k\}_{k=1}^{K}$ are a function of the symbols $\{j_m\}_{i=-M+1}^{M-1}$.

Since the jammer is unaware of the time delay introduced, it treats the timing offset as a uniform random variable in the interval $\tau \in [0, T)$. Under such scenarios, the average $p_e$ of a M-QAM victim signal along any signaling dimension that the jammer intends to maximize is given by

$$p_{e,\tau,PAM}(\widehat{j}, P_S, P_J) = \left(1 - \frac{1}{\sqrt{M}}\right)\frac{1}{2}\sum_{j_{-M+1}} \cdots \sum_{j_{M-1}} \left[\mathsf{erfc}\left(\frac{\sqrt{\mathrm{SNR}}\frac{d_{\min}}{2} + \sqrt{\mathrm{JNR}}\widehat{j}}{\sqrt{2}}\right) + \mathsf{erfc}\left(\frac{\sqrt{\mathrm{SNR}}\frac{d_{\min}}{2} - \sqrt{\mathrm{JNR}}\widehat{j}}{\sqrt{2}}\right)\right],$$

where averaging is performed over all the ISI terms in order to evaluate the error rate and $\widehat{j}$ indicates either the in-phase part or the quadrature part of $\sum_{m=-M+1}^{M-1} \bar{j}_m \widehat{g}(t - mT - \tau)$. The optimization problem that the jammer must solve to obtain the optimal jamming signal distribution is given by

$$\max_{f_{\bar{J}}} E_{f_{\bar{J}}}\left[E_\tau\left(p_{e,\tau,PAM}\left(\widehat{j}, P_S, P_J\right)\right)\right] \text{ s.t. } E(||\bar{j}||^2) \leq 1, \quad (16)$$

where $f_{\bar{J}}$ indicates the jamming signal distribution. Even in this scenario, the optimal jamming signal distribution can be shown to have two signal levels along any signaling dimension. However, it can be seen from (14)-(16) that the optimization problem in this scenario is more complicated compared to the coherent and non-coherent scenarios due to the inter-symbol interference (ISI) introduced by the incorrect sampling time offset. Hence, in this section, we study the performance of the optimal jamming signal obtained in Section III in the presence of a random time delay introduced by the wireless channel.

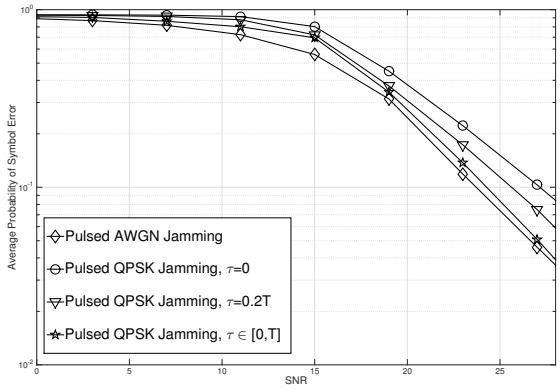


Fig. 3. Comparison of jamming techniques against a 16-QAM victim signal in the presence of timing synchronization errors, $JNR = 10\ dB$.

Fig. 3 shows the error rate performance of the optimal jamming signal in Table III-B against a 16-QAM victim signal in the presence of time synchronization errors ($M = 2$ and a roll-off factor of $0.65$ for the raised cosine pulse shape). It is seen that the $p_e$ achieved by the jamming signal in Table III-B is lower as compared to the perfectly synchronized scenario because of the ISI. However, it is seen that the $p_e$ achieved by the jamming signal in Table III-B is still higher than the $p_e$ achieved by the pulsed-AWGN jamming. Although there may be specific cases where ISI helps the jammer to increase the error rates by causing constructive interference against the victim signal, on an average the random time delay $\tau \in [0, T]$ only reduces the impact of jamming (remember that the modulation-based jamming signal can still result in a linear decay of the error rate with respect to SNR). This is because

[4]Even in the non-coherent scenario, BPSK continues to be the optimal jamming signal against any $M$-PAM signal. This is because a) only the projections of the received signal along the in-phase dimension are used to decode the victim signal and b) QPSK is a sub-optimal jamming signal because energy is wasted by transmitting along the quadrature dimension.

the timing offset essentially creates a multiple level jamming signal (the overall effect of ISI translates into this multi-level effect) which as we proved earlier is sub-optimal because a two-level signal is the optimal jamming signal distribution. This explains the lower error rates achieved in comparison to a perfectly synchronous case and seems to approach the performance of an AWGN jamming signal (as seen in Fig. 3 when $\tau \in [0, T]$)

### C. Signal Level Mismatch

When the jammer is not perfectly aware of the power levels of the communication and the jamming signals i.e., $P_S$ and $P_J$ at the victim receiver, the optimization problems presented before in Section III do not result in the optimal jamming signal distribution. Due to this uncertainty, the error rate performance of the jamming signals shown in Table III-B will be degraded. However, if the uncertainty in the knowledge of $P_S$ or $P_J$ is accounted for, then the jammer's performance can be significantly improved as will be shown below. For ease of analysis, we assume that the jammer is not aware of $P_S$ exactly. Notice that the error in the knowledge of $P_J$ can also be accounted for along similar lines. If $\epsilon$ is the error in the jammer's knowledge about $P_S$, then the jammer assumes that the received signal at the victim receiver is given by

$$\bar{y}_k = \sqrt{P_S + \epsilon}\bar{s}_k + \sqrt{P_J}\bar{j}_k + \bar{n}_k, \ k = 0, 1, \ldots, K. \quad (17)$$

To obtain the optimal jamming signal distribution under such scenarios, the jammer solves the following optimization problem;

$$\max_{f_{\bar{j}}} \ E_{f_{\bar{j}}}\Big[E_{\epsilon}\Big(p_e\left(\bar{j}, P_S + \epsilon, P_J\right)\Big)\Big] \ \text{s.t.} \ E(||\bar{j}||^2) \leq 1. \quad (18)$$

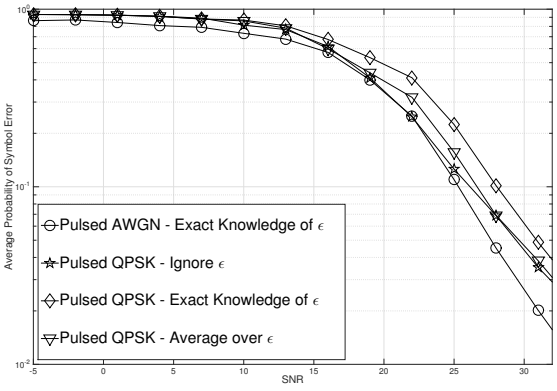


Fig. 4. Comparison of jamming techniques against a 16-QAM victim signal in the presence of signal level mismatch, $JNR = 10\ dB$.

Notice that this optimization problem is similar to the formulation in a phase-asynchronous scenario shown in (12). Thus, following the same principles as earlier, the optimal jamming signal can be shown to have a two-level distribution along any signaling dimension. As before, we use the Matlab's optimization toolbox to find the optimal jamming signal distribution across all ranges of $SNR$ and $JNR$. Fig. 4 shows the performance of the optimal jamming signal in comparison to cases when the jammer is perfectly aware of $\epsilon$ and can therefore perfectly evaluate the optimal jamming signal distribution at any given time instant. Also shown are the performance of the pulsed AWGN and pulsed QPSK jamming signals when the jammer accounts for or ignores the error $\epsilon$. Specifically, in Fig. 4, the error $\epsilon$ was taken to be distributed as zero mean Gaussian with variance proportional to the signal power $P_S$ and $\text{SNR} = P_S/\sigma^2$. It is seen that when the error is accounted for, the jammer can still perform better than a) a pulsed QPSK jamming signal that ignored $\epsilon$ and b) the naive AWGN jamming signal.

## V. Jamming an OFDM Signal

In this section, we study the optimal jamming signal distribution against OFDM-modulated wireless signals. The jammer can easily identify whether the victim is using a single-carrier or an OFDM-modulated signal by employing simple feature-based classification techniques [25]. Under such scenarios, we show that the optimal jamming signal is also an OFDM modulated signal i.e., jamming in the frequency domain (by this we mean that the jammer's symbols are modulated onto the sub carriers of the OFDM signal) is more effective in comparison to jamming in the time domain.

Typically, most earlier literature considered AWGN jamming signals to attack the victim (see the tutorial paper [17] for more information), but as we show below this is sub-optimal. This follows directly from the results presented in the previous sections in the context of a single carrier system. By using the optimal jamming signals obtained in this paper, innovative power efficient jamming techniques such as cyclic prefix jamming, preamble jamming among others can be performed. However, this is not the major concern of this paper. In other words, we are concerned with jamming data only and not control or synchronization parts of the victim's transmission. For more information on efficient OFDM jamming attacks, please refer to [17].

In an OFDM system, an IFFT operation converts the frequency domain modulated symbols to time domain signals before they are transmitted. The time domain OFDM symbol $s(l)$ is given by

$$s(l) = \sum_{k=1}^{N_{sc}} S(k) \exp(j2\pi lk/N_{sc}), \ \forall 0 \leq l \leq N_{sc} - 1, \quad (19)$$

where $S(k)$ is the frequency domain signal/symbol which is typically a digital amplitude phase-modulated wireless signal, such as $M$-QAM and $N_{sc}$ are the total number of sub-carriers used in the OFDM transmission. Assuming that the victim signal passes through an AWGN channel (received power is constant over the observation interval) while being attacked by a time-domain jamming signal $j(l)$, the received signal at the victim receiver is given by

$$y(l) = \sqrt{P_S}s(l) + \sqrt{P_J}j(l) + n(l), \quad (20)$$

where $P_S, P_J$ hold the same meaning as earlier. At the receiver, the OFDM signal is passed through an FFT block to recover the underlying frequency domain signal $S(k)$ as

$$Y(k) = \sum_{l=0}^{N_{sc}-1} y(l) \exp(-j2\pi kl/N_{sc})$$

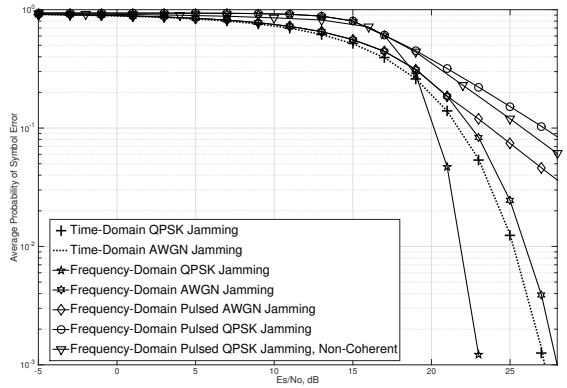


Fig. 5. Comparison of jamming techniques against a OFDM-modulated 16-QAM victim signal, $JNR = 10\ dB$.

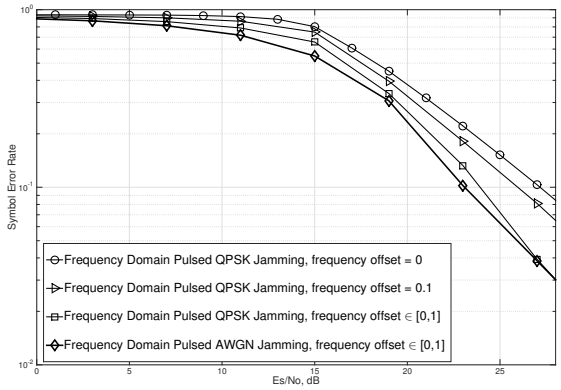


Fig. 6. Comparison of jamming techniques against a OFDM-modulated 16-QAM victim signal in the presence of a frequency offset, $JNR = 10\ dB$.

$$= \sqrt{P_S}S(k) + \sqrt{P_J}\sum_{l=0}^{N_{sc}-1} j(l)\exp(-j2\pi kl/N_{sc}) + N(k), \quad \forall 0 \leq k \leq N_{sc}-1. \tag{21}$$

Notice that (21) is similar to the single carrier scenario discussed before in the Section III. For instance, when $S(k)$ is a $M$-QAM modulated victim signal, then based on the analysis in Sections III and IV, the optimal jamming signal $J(k) = \sum_{l=0}^{N_{sc}-1} j(l)\exp(-j2\pi kl/N_{sc})$ is a pulsed-QPSK signal. Therefore, this leads to the conclusion that the optimal time-domain jamming signal $j(l)$ is OFDM-modulated with frequency-domain symbols $J(k)$, whose distribution was obtained earlier in Sections III, IV. Thus, all the findings of the earlier sections in the context of a single carrier system can also be extended to the case of OFDM signaling.

Fig. 5 shows the performance of the various jamming signals against OFDM-based 16-QAM modulated victim signal. Specifically, each OFDM symbol consists of $64$ subcarriers with data modulated on only $52$ subcarriers. Further, a cyclic prefix of length $16$ was appended to each OFDM symbol in the simulations. Notice that the time-domain AWGN and QPSK jamming signals achieve the same error rates because the QPSK signal will also look similar to an AWGN signal after the FFT operation at the victim receiver. It is also seen that frequency domain jamming (with the same structure as the victim signal i.e., $64$ subcarriers with data modulated on only $52$ subcarriers and a cyclic prefix of length $16$) performs significantly better than time domain jamming and also the frequency domain pulsed QPSK jamming signal achieves a higher error rate than all other jamming signals. These findings are in agreement with the results discussed earlier in Section III. Also shown is the performance of the pulsed QPSK jammer when the victim signal and the jamming signal are non-coherent i.e., there is a phase-offset between these signals at the victim receiver. In this case, the optimal jamming signal is obtained by following the steps in Section IV-A. The performance of the optimal jamming signal even in this case is similar to that of the single carrier system.

It is well-known that frequency offset is a major problem in OFDM signals in comparison to the timing offset. This is because the presence of a cyclic prefix helps overcome the effects of a timing offset introduced by the wireless channel. Moreover the effects of the frequency offset in OFDM-based signals can be related to the effects of a timing offset in single carrier systems. Therefore, the jamming performance in the presence of a frequency offset between the victim and the jamming signals can be related to the timing asynchronous case in a single carrier system which was discussed before in Section IV-B. A frequency offset between the jamming and the victim signal leads to inter-carrier interference (ICI), which similar to ISI in a single carrier system, degrades the performance of the jamming signal with respect to the error rates created at the victim receiver. Since optimizing the jamming signal in the presence of a frequency/timing offset is complex due to the presence of ICI/ISI, we study the performance of the jamming signal obtained in the coherent scenario when it is used in the presence of a frequency offset.

When the jamming and the victim signals are off by an integer number of subcarrier spacings, the orthogonality between the subcarrier's of the jammer and the victim signals remains intact due to the cyclic shift created by the receiver's FFT operation. Thus no ICI is created in this case and the performance would be similar to that shown in Fig. 5. Hence, only a fractional frequency offset, i.e., offsets less than a subcarrier spacing, are considered in this analysis. Fig. 6 shows the error rates achieved by the jamming signal when the jamming and the victim signal are frequency asynchronous. As expected, the error rate performance is degraded in comparison to the case when the jamming and the victim signals are perfectly aligned in terms of the frequency. Note that the performance of the AWGN jamming signal will also be affected by a frequency offset unlike the case when their performance is not impacted by a phase offset between the jamming and the victim signals.

## VI. The case of Multiple Jammers

In this section, we analyze the error rates achieved by multiple coordinating jammers attacking a single victim transmitter-receiver pair. It is assumed that the total average power available with the multiple jammers is the same as the average power available with a single jammer that was studied in the previous sections. When the jammers are coordinated, they can jointly evaluate the optimal joint jamming signal distribution that maximizes the probability of error at the victim receiver. We show the superior performance of such a joint

optimal jamming signal distribution by comparing it against the cases where multiple jammers are un-coordinated and hence employ the optimal single jamming signal distributions obtained in previous sections. Further, these increased error rates can be achieved by using only lower average power levels at each jamming node which also helps in reducing the jammer detection probability. The assumption that the jammers can coordinate enables us to evaluate the maximum possible error rates that can be created at the victim receiver under a given average power constraint. In scenarios, where such coordination is not possible, alternate techniques such as online learning [26]-[28] can be employed. In this paper, we restrict to the cases where coordination between the multiple jammers is possible, for example by communicating via a side channel.

The received signal at the victim receiver when $N$ coherent and synchronous jammers attack a single victim signal $s$, is given by

$$\bar{y}_k = \sqrt{P_s}\bar{s}_k + \sum_{i=1}^{N} \sqrt{P_J(i)}\bar{j}_{i,k} + \bar{n}_k,\ k = 1, 2, \ldots, K \quad (22)$$

where $\bar{y}_k, \bar{s}_k, \bar{n}_k$ were defined earlier and $\bar{j}_{i,k}$ indicates the jamming $k$th jamming symbol sent by the $i$th jammer such that $E(||\bar{j}_i||^2) \leq 1$. Here, $P_J(i)$ is the power of the $i$th jamming signal at the victim receiver and $\sum_i P_J(i) = P_J$, i.e., the total average power available with the multiple jammers is the same as available with a single jammer considered in the previous sections. For ease of exposition, we show the analysis for the optimal jamming signal along any one signaling dimension which can be easily extended to any dimension i.e., to consider any standard modulation schemes following the analysis done in the previous sections.

By letting $j_i$ indicate the in-phase or the quadrature parts of the jamming signal, the $p_e$ along any signaling dimension for an $M$-QAM signal, is given by (4) by replacing $j$ in (4) with $\sum_{i=1}^{N} \sqrt{P_J(i)} j_i$. Since $j_i$ indicates either the in-phase or the quadrature components of the jamming signal, we have the following average power constraint $E(j_i^2) \leq \frac{1}{2}$. Hence, when the jammers can coordinate and try to maximize the error rate at the victim receiver, they solve the following optimization problem to obtain the optimal jamming signal along any signaling dimension,

$$\max E(p_e(j_1, \ldots, j_N))\ s.t.\ E(j_n^2) \leq \frac{1}{2},\ n = 1, 2, \ldots, N.$$

where the expectation is taken with respect to $f_{j_1,j_2,\ldots,j_N}(j_1, \ldots, j_N)$, which denotes the joint optimal jamming signal distribution of all the jammers that intend to attack the victim.

***Remark** 7:* When the jammer's cannot coordinate with each other, then the joint jamming signal distribution along any one signaling dimension $f_{j_1,j_2,\ldots,j_N}(j_1, \ldots, j_N)$ is given by $\prod_{i=1}^{N} f_{J_i}(j_i)$, where the jamming signal distribution for the $i$th jammer $f_{J_i}(j_i)$, is given by the optimal single jammer distribution derived earlier in Sections III-V. We show that this results in a sub-optimal jamming performance when compared to scenarios where the jammer's can coordinate.

The following theorem establishes the structure of the joint optimal jamming signal distribution $f_{j_1,j_2,\ldots,j_N}(j_1, \ldots, j_N)$ along any signaling dimension.

***Theorem** 4:* The joint optimum jamming signal distribution $f_{j_1,j_2,\ldots,j_N}(j_1, \ldots, j_N)$ along any signaling dimension, when $N$ jammers attack a single victim transmitter-receiver pair is defined by $N + 1$ levels.

*Proof:* Define set U as

$$U = \Big\{(u_1, u_2, u_3, \ldots, u_{N+1}) : u_1 = p_e(j_1, j_2, \ldots, j_N), \\ u_{n+1} = j_n^2,\ \forall n = 1, 2, \ldots, N\Big\} \quad (23)$$

Since $p_e$ is continuous, the mapping defined by $(p_e(j_1, j_2, \ldots, j_N), j_1^2, j_2^2, \ldots, j_N^2)$ is also continuous in the domain $|j_n| \leq j_{\max},\ \forall n = 1, 2, \ldots, N$, where $j_{\max}$ indicates the maximum signal level that can be sent by the jammers. Such an assumption is common for wireless communication systems because arbitrarily large signal levels cannot be generated by practical transmitters.

Now, $U$ is a compact set, because the continuous image of a compact set is also compact. Let $V$ represent the convex hull of $U$. Since $U$ is compact, $V$ is closed with dimensions not exceeding $N + 1$ as it is a subset of $(\mathbb{R}^+)^{N+1}$.

Now, define a set $W$ as

$$W = \Big\{(w_1, w_2, w_3, \ldots, W_{N+1}) : \\ w_1 = \int\int p_e(j_1, j_2, \ldots, j_N) f_{j_1,j_2,\ldots,j_N} dj_1 dj_2 \ldots dj_N, \\ w_{n+1} = \int\int j_n^2 f_{j_1,j_2,\ldots,j_N} dj_1 dj_2 \ldots dj_N, \\ \forall n = 1, 2, \ldots, N\Big\}, \quad (24)$$

where $f_{j_1,j_2,\ldots,j_N}(j_1, \ldots, j_N)$ is the joint pdf of the jamming signals used by the coordinated jammers. Notice that the elements of $W$ are the expected values of the elements of $U$, where expectation is taken with respect to the joint jamming signal distribution $f_{j_1,j_2,\ldots,j_N}$.

It is known from previous results that, if a random variable $\Theta$ takes values in a set $\Omega$, then its expected value $E(\Theta)$ takes values in the convex hull of $\Omega$ [20], [29, Appendix 4.B]. This indicates that $W$ is in the convex hull $V$ of the set $U$. In other words, we have $W \subseteq V$.

We will now show that $V \subseteq W$. Since V is the convex hull of U, each element inside $V$ can be easily expressed as $\mathbf{v} = \sum_{\ell=1}^{L} \lambda_\ell (p_e(j_1^{(\ell)}, j_2^{(\ell)}, \ldots, j_N^{(\ell)}), j_1^{(\ell)}, j_2^{(\ell)}, \ldots, j_N^{(l)})$ with $\sum_\ell \lambda_\ell = 1$ and $\lambda_\ell > 0$. Here $j_i^{(\ell)}$ indicates the $\ell$th point of the the jamming signal $j_i$. See that set $W$ has an element equal to $\mathbf{v}$ for $f_{j_1,j_2,\ldots,j_N} = \sum_{\ell=1}^{L} \lambda_\ell \delta(\mathbf{j} - \mathbf{j}^{(\ell)})$ where $\mathbf{j} = (j_1, j_2, \ldots, j_N)$ and $\mathbf{j}^{(\ell)} = (j_1^{(\ell)}, j_2^{(\ell)}, \ldots, j_N^{(\ell)})$. Therefore, every element of $V$ is also a subset of $W$ which leads to $V \subseteq W$. Using the above two results we have, $V = W$.

By using the Caratheodory's theorem [18], we have that any point inside $V$ or $W$ can be expressed as a convex combination of *at most* $L = N + 2$ points that belong to $U$. Now since the jammers intend to maximize $p_e$, the optimal value lies on the boundary of $V$ which can be expressed by at most $N + 1$ elements that belong to $U$. Thus the optimal pdf $f_{j_1,j_2,\ldots,j_N}$ is

described by $N+1$ vectors namely $\{(j_1^{(\ell)}, j_2^{(\ell)}, \ldots, j_N^{(\ell)})\}_{\ell=1}^{N+1}$ such that

$$\sum_{\ell=1}^{N+1} \lambda_\ell (j_n^{(\ell)})^2 \leq \frac{1}{2}, \ \forall n = 1, 2, \ldots, N. \tag{25}$$

This concludes the proof of the Theorem.

***Remark** 8:* When a single jammer attacks a victim receiver, i.e., the case of $N = 1$ given in (22), then Theorem 4 gives the result mentioned earlier in Section III-A regarding the optimality of a two-level jamming signal distribution along any signaling dimension.

By following the procedure in Section III, and using the structure of the optimal jamming signal distribution in Theorem 4, we obtain the error rates against standard digital amplitude phase modulated victim signals. By following the analysis in Section IV, the optimum jamming signal distribution in the presence of non-idealities in the channel can also be obtained. Numerical results that show the performance of the multiple jamming signals is discussed next.

### A. Results

Fig. 7 shows the error rates achieved by various jamming techniques when $N$ jammers attack a single victim node. In all the cases, the overall average power used by the jammers is restricted to $P_J$ in order to compare with the error rate performance of a single jammer and each jammer has equal power equal to $\frac{P_J}{N}$. It is clearly seen that unless both the jammer's coordinate, the gains in $p_e$ are not achievable. By coordinate, we mean that the jammers should pulse at the same time when pulsing is optimal i.e., all the jammers should transmit at the same time by employing the optimal joint jamming signal distribution (which in this paper is obtained by using the optimization toolbox in Matlab). It is worth noticing that a $3dB$ SNR gain is achieved when the number of jammers is doubled i.e., $3dB$ higher SNR is required at the victim to attain the same error rate.

When coordination is not possible, the multiple jammer performance is limited and it can only achieve the error rates achieved by a single jammer. Notice that in the ranges where QPSK jamming is optimal for a single jammer, the error rates achieved by two un-coordinated jammers is degraded. This is because a positive signal sent by one jammer can be cancelled by a negative signal that may be sent by the other un-coordinated jammer. However when the jammers are pulsing, the probability that their pulsing instants may match is on the order of $\rho^2$ ($\rho$ being the pulse jamming probability) which is small, and hence there is a small probability that their signals may cancel each other. This is why the error rates achieved by the un-coordinated jammers in the pulsing region matches the error rate achieved by a single jammer. As expected, the performance of all these jamming signals is better than the naive AWGN jamming signal.[5] Also shown is the performance of the optimum jamming signal distribution when the jammers are non-coherent (phase offset between the signals) when their signals reach the victim receiver. As expected the performance of the joint optimal non-coherent jamming signal is degraded in comparison to the coherent case.

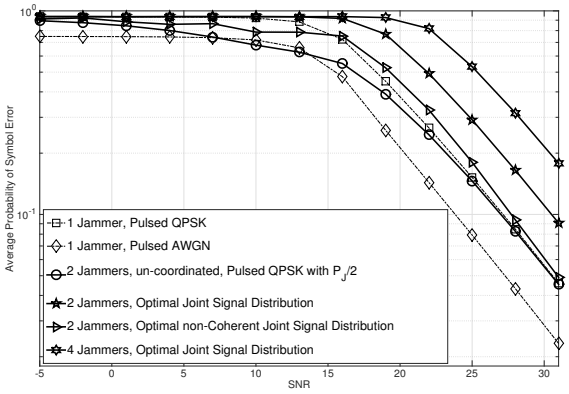


Fig. 7. Comparison of jamming techniques when multiple jammers attack a single 16-QAM modulated victim signal, JNR=10dB.

## VII. Conclusion

In this paper, we characterized the optimal statistical distribution for power-constrained jamming signals that jam digital amplitude-phase modulated constellations in an AWGN channel in both single carrier and OFDM-based wireless systems. The analysis in this paper shows that modulation-based pulsed jamming signals are optimal in both coherent and non-coherent (phase asynchronous) scenarios. As opposed to the common belief that matching the victim signal (correlated jamming) increases confusion at the victim receiver, our analysis shows that the optimal jamming signals match standard modulation formats only in a certain range of signal and jamming powers. Beyond this range, either binary or quaternary pulsed jamming is the optimal jamming signal. An interesting relationship between these optimal jamming signals and the well-known pulse jamming signals discussed in the context of spread spectrum communications was illustrated. As expected, the performance of these optimal jamming signals was seen to be degraded when the victim and the jamming signals are not phase or time synchronous or when it does not have perfect knowledge of the power levels of the victim and the jamming signals although the optimal jamming signal distributions don't change. Against OFDM-based signaling, it was observed that OFDM-based jamming signals that use the optimal jamming signal distributions obtained in single carrier scenarios are optimal and that their performance is degraded in the presence of channel non-idealities such as residual carrier frequency offset. Upon extending this analysis to a multiple jammer scenario, it was found that gains in terms of the error rate at the victim receiver are possible only when the multiple jammers are perfectly coordinated, otherwise multiple jammers can only match the impact of a single jammer. The optimal jamming signal distributions against practical wireless signals that employ error correction coding techniques is being investigated.

[5]Note, that the performance of multiple pulsed AWGN jammers will coincide with the performance of a single pulsed AWGN jamming signal.

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